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# Algebra of Effects in the Formalism of Quantum Mechanics on Phase Space as an M. V. and a Heyting Effect Algebra*

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We prove that the algebra of effects in the phase space formalism of quantum mechanics forms an M. V. effect algebra and moreover a Heyting effect algebra. It contains no nontrivial projections. We equip this algebra with certain nontrivial projections by passing to the limit of the quantum expectation with respect to any density operator.

**KEY WORDS:** M. V. algebra; quantum mechanics on phase space; completion of M. V. algebras.

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## 1. INTRODUCTION

An effect in quantum mechanics on a Hilbert space $\mathfrak{H}$ is an operator $A$ in $\mathfrak{H}$ such that $0 \leq A \leq I$, and two effects, $A$ and $B$, are added to get $A \oplus B$ iff $0 \leq A \oplus B \leq I$, where $A \oplus B$ is the operator addition. We wish to specialize this concept to the effects that arise naturally in the formalism of quantum mechanics on phase space (Schroeck, 1996). From this, we will have an immediate generalization to $H = L^2(\Gamma, d\mu)$ where $\Gamma$ is any space and $d\mu$ is a (group-)invariant measure on $\Gamma$. Then we wish to see what axioms are satisfied by this "algebra" of effects.

Our plan is to first review the formalism of quantum mechanics on phase space, then to define the algebra of effects within this formalism, and then to show that we in fact have an M. V. effect algebra and a Heyting effect algebra. Finally, we show that from the algebra of effects, which has no nontrivial projections in it, we may obtain certain nontrivial projections by taking a limit of the expectation values of the effects.

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## 2. THE FORMALISM OF QUANTUM MECHANICS ON PHASE SPACE

In the formalism of quantum mechanics on phase space (Schroeck, 1996), we begin with a given kinematical locally compact Lie group, $\mathcal{G}$, with Lie algebra $\mathfrak{g}$ that is finitely generated. We have in mind taking for $\mathcal{G}$ the Heisenberg group, the Galilei group, the Poincaré group, or even the affine group, etc. (The rest of this section is a brief review.) Then we find all the symplectic spaces of $\mathcal{G}$. This is carried out by a bit of argument from Lie group cohomology (Guillemin and Sternberg, 1991; Schroeck, 1996): Let $\omega$ be an element of the 2-cocycle of $\mathfrak{g}$, $Z^2(\mathfrak{g})$. Let $\mathfrak{h} = \{y \in \mathfrak{g} \mid \omega(y, x) = 0 \text{ for all } x \in \mathfrak{g}\}$. $\mathfrak{h}$ is a Lie sub-algebra of $\mathfrak{g}$. Form the subgroup $\mathcal{H} = \exp\{\mathfrak{h}\}$. If $\mathcal{H}$ is a closed subgroup of $\mathcal{G}$, then we define $\Gamma = \mathcal{G}/\mathcal{H}$, a (typical) symplectic space with respect to $\mathcal{G}$. We have the $\mathcal{G}$-invariant measure $d\mu$ coming directly from $\omega$. Every symplectic $\mathcal{G}$-space is of the form of $\Gamma$ or a direct sum of such $\Gamma$'s.

$\Gamma$ being a symplectic space, we may refer to it as a phase space in the terminology of physics (Guillemin and Sternberg, 1991). The group elements of $\mathcal{G}$ being just the displacements in $\Gamma$, we have an immediate interpretation of the generators of $\mathcal{G}$ (the Lie algebra $\mathfrak{g}$) as the momentum operators, the position operators, the rotation or spin operators, the dilation operators, etc.

Let $\mathfrak{H}$ denote an irreducible representation space for $\mathcal{G}$, and denote the representation $U(g)$, $g \in \mathcal{G}$. Let $\sigma : \mathcal{G}/\mathcal{H} \to \mathcal{G}$ be a Borel section. ($\sigma$ is continuous for all the representations of the kinematical groups.) What comes as a surprise is that, almost always, we may intertwine $\mathfrak{H}$ with a subspace of $L^2(\Gamma, d\mu)$ in the following fashion: Take a fixed $\eta \in \mathfrak{H}$, $\| \eta \| = 1$, and form $[W^\eta(\psi)](g) \equiv \langle U(g)\eta, \psi\rangle_{\mathfrak{H}}$, $\psi \in \mathfrak{H}$. Then take $[W^\eta(\psi)](\sigma(x))$ for $x \in \Gamma$. In this way, we think of $\Gamma$ as being embedded in $\mathcal{G}$. For a certain class of the $\eta$'s defined below, we find that $W^\eta$ is an isometry into a subspace of $L^2(\Gamma, d\mu)$. In fact, there are an infinite number of such $\eta$'s and subspaces of $L^2(\Gamma, d\mu)$ of the form $P^\eta L^2(\Gamma, d\mu) = W^\eta \mathfrak{H}$, with $P^\eta$ the projection onto the subspace. The spaces $P^\eta L^2(\Gamma, d\mu) = W^\eta \mathfrak{H}$ are known to be orthogonal for the $\eta$'s orthogonal in a certain inner product, due to the so-called Orthogonality Theorem (Schroeck, 1996), since $\mathcal{G}$ is locally compact. The precise condition on the vector $\eta$ is that $\langle U(\sigma(x))\eta, \eta\rangle_{\mathfrak{H}}$ must be square integrable with respect to $d\mu$, and that $U(g)\eta = \alpha(g)\eta$, $\alpha(g) \in \mathbb{C}$, $| \alpha(g) | = 1$, for all $g \in \mathcal{H}$. These are called the admissibility condition and the $\alpha-$admissibility condition, respectively (Schroeck, 1996).

This $\mathfrak{H}$ is a quantum mechanical representation space, the elements $\psi \in \mathfrak{H}$ are the wave functions, etc. An observable on $\mathfrak{H}$ is (taken to be) a self-adjoint operator on $\mathfrak{H}$ that may be generated from the elements of the concrete Lie algebra. (Say, take $S_x$, $S_y$, $S_z$ if the system has spin, take $P_x$, $P_y$, $P_z$, $Q_x$, $Q_y$, $Q_z$ if the system has momentum and position, etc.) Consider $\mathfrak{g}/\mathfrak{h}$ and take the extension $\sigma(\mathfrak{g}/\mathfrak{h}) \to \mathfrak{g}$. Let $\sigma(\mathfrak{g}/\mathfrak{h}) = \text{span}\,\{i\ B_1, iB_2, \ldots, iB_n\}$. Form the (connected) Lie group by obtaining

$$W(x_1, \ldots, x_n) = W(x) = \exp\{i(x_1 B_1 + \cdots + x_n B_n)\}, x_i \in \mathbb{R}.$$

$W(x)$ is a unitary operator in $\mathfrak{H}$ since $B_i$ is self-adjoint for each $i$. Thus we have, abusing the notation, $\sigma([g]) = x$ and $U([g]) = W(x)$.

The observables are associated with a positive operator valued measure as follows:

$$T^\eta(x) = \mid W(x)\eta\rangle\langle W(x)\eta \mid, \quad x \in \Gamma.$$

Note the important property

$$T^\eta(x)T^\eta(y) \neq 0, \quad \text{for} \quad x \neq y,$$

since

$$\langle W(x)\eta \mid W(y)\eta\rangle \neq 0 \quad \text{for} \quad x \neq y$$

in general.

A physical interpretation: The $\eta$ is fixed, and may satisfy additional constraints to get special properties for the $T^\eta$s. For an interpretation, if $\psi$ is a vector in $\mathfrak{H}$, then

$$Tr(P_\psi T^\eta(x)) = \langle \psi, W(x)\eta\rangle\langle W(x)\eta, \psi\rangle$$

is the transition probability to the vector state given by $W(x)\eta$. We are choosing an $\eta$ and then translating it all over, thereby getting an interpretation of what we are doing here. The fact that $\langle W(x)\eta \mid W(y)\eta\rangle \neq 0$ is just a result of the nonlocal nature of our physical interpretation.

If we now define

$$T^\eta(\Delta) = \int_\Delta T^\eta(x)d\mu(x), \ \Delta \text{ a Borel set in} \Gamma,$$

then $\Delta \to T^\eta(\Delta)$ is a positive operator valued measure (Schroeck, 1996) called a localization operator for the phase space $\Gamma$. Next, we take

$$A^\eta(f) = \int_\Gamma f(x)T^\eta(x)d\mu(x)$$

for any $\mu$-measurable function $f$ on $\Gamma$. Note that $A^\eta(\chi_\Delta) = T^\eta(\Delta)$. Also note that $A^\eta(1) = I$. From this, we get

$$\| A^\eta(f) \| \leq \operatorname{ess}\sup_{x\in\Gamma} \mid f(x) \mid$$

and

$$0 \leq A^\eta(f) \leq I \quad \text{for} \quad 0 \leq f(x) \leq 1, \text{a.e. } x.$$

Note that we have that $A^\eta(f) = A^\eta(h)$ as operators iff $f - h$ is of $\mu$-measure zero. Later, we will relax this.

A function of the observable $B_1$ in the Lie algebra is given by

$$A^\eta(f) = \int_\Gamma f(x_1) T^\eta(x) d\mu(x),$$

$f$ a function of $x_1$ only. Similarly for $B_i$. In fact, for each

$$B_i = c_i \int_\Gamma x_i T^\eta(x) d\mu(x), \quad \text{for some} \quad 0 < c_i \leq 1,$$

and for each measurable function $f$ on $B_i$, there is a function $\mathcal{F}$ such that

$$f(B_i) = \int_\Gamma \mathcal{F}(x_i) T^\eta(x) d\mu(x) = A^\eta(\mathcal{F}).$$

(See the proof in (Schroeck, 1996). It depends on the "informational completeness" of the $A^\eta$—that we can distinguish between all states $\rho$ with the $\{Tr(\rho A^\eta(f)) \mid f$ is $\mu$-measurable$\}$. This is *equivalent* to the condition on $\eta$ that $< W(x)\eta \mid W(y)\eta > \neq 0$ for $x \neq y$. The proof holds in the sense of being in the limit of the $A^\eta(\mathcal{F})$.) Note that if $f(B_i) = E^{B_i}(\Delta_i)$, then $f(B_i) \neq A^\eta(\chi_{I\times\Delta_i\times I})$.

*Definition 2.1.* We define the algebra of effects in the formalism of quantum mechanics on phase space as the set $\mathcal{E} \equiv \{A^\eta(f) \mid f$ is measurable, $0 \leq f(x) \leq 1$ a.e. $x\}$. Then define

$$A^\eta(f) \oplus A^\eta(h) = A^\eta(f) + A^\eta(h) = A^\eta(f + h), \tag{2.1}$$

which, in turn, is such an effect if and only if $(f + h)(x) \leq 1$, a.e. $x$.

We now have the following:

$$Tr(\rho A^\eta(f)) = \int_\Gamma f(x) Tr(\rho T^\eta(x)) d\mu(x) = \int_\Gamma f(x) \sum_i d_i Tr(P_{\psi_i} T^\eta(x)) d\mu(x)$$

where $\rho = \sum d_i P_{\psi_i}$ is a general density matrix. We have the interpretation that when we measure $A^\eta(\chi_\Delta)$ in density state $\rho$ we get the probability that $\rho$ will be in the state $P_{W(x)\eta}$ for some $x \in \Delta$.

Note that what we will prove about the algebra of effects in our formalism is, in fact, true about any image in any Hilbert space of the set of fuzzy sets! The setting of the formalism of quantum mechanics in phase space is just an example.

## 3. PHYSICAL EXAMPLES

For our first example, we take $\mathfrak{H} = L^2(\mathcal{S}^2)$ for a system with spin-$\frac{1}{2}$, and $A = S_z^+$, $B = S^+_{(z+x)/2^{1/2}}$ where $S_u^\pm$ is the projection onto the set of vector states, $\psi$, such that $S_u^\pm \psi = \psi$. See (Schroeck, 1982 and 1996. Chap. II.3.A.). Note that

$\{S_u^+, S_u^-\}$ is a projection valued measure, and thus contains only effects. In $L^2(\mathcal{S}^2)$ there is only one form of a nontrivial projection, namely

$$T(x) = \frac{1}{2}(I + x \cdot \sigma), x \in \mathbb{R}^3, \parallel x \parallel = 1, \sigma = (\sigma_1, \sigma_2, \sigma_3)$$

a representation of the Pauli spin algebra. Thus, the only freedom in choosing an $\eta$ is in the direction it "points" on the unit sphere. We have $\mathcal{S}^2$ as a homogeneous space of the rotation group. With the invariant measure $\mu$ on $\mathcal{S}^2$, normalized so that $\mu(\mathcal{S}^2) = 2$, and with $\eta$ chosen to point in the direction of the North Pole, then we have

$$A^\eta(1) = \int_{\mathcal{S}^2} T(x) d\mu(x) = I,$$

and

$$A^\eta(x_j) = \int_{\mathcal{S}^2} x_j T(x) d\mu(x) = \int_{\mathcal{S}^2} x_j \frac{1}{2}(I + x \cdot \sigma) d\mu(x) = \frac{1}{3}\sigma_j.$$

Hence, we get all the generators $i\sigma_j$ of the rotation group on $\mathcal{S}^2$. Furthermore, we can get all effects in the set $\{A^\eta(f) \mid f \text{ is measurable}\}$. The set is informationally complete in $L^2(\mathcal{S}^2)$.

For our second example, take $\mathfrak{H} = L^2(\mathbb{R})$, $A = E^P(\Delta_1)$, $E^P$ being the spectral measure for $P$, the momentum, $\Delta_1$ a Borel set in $\mathbb{R}$, and $B = E^Q(\Delta_2)$, $E^Q$ being the spectral measure for the position, $Q$, etc. (Note: $E(\Delta) = \int_\Delta dE_\lambda$.) Thus, these two operators do not commute either, but they are effects, since they are both projections. Then $\{iP, iQ, iI\}$ are generating elements of the Lie algebra for the Heisenberg group. Now take $\eta$ to stand for a vector state that has

$$\langle \eta, P\eta \rangle = 0, \quad \langle \eta, Q\eta \rangle = 0.$$

Then, for $\sigma(p, q) = (p, q, 0)$,

$$\langle W(\sigma(p,q))\eta, PW(\sigma(p,q))\eta \rangle = p \quad \text{and} \quad \langle W(\sigma(p,q))\eta, QW(\sigma(p,q))\eta \rangle = q;$$

that is, $T^\eta(p, q)$ is the state that corresponds to moving $\eta$ by $(p, q)$ in the phase space. $A^\eta(\chi_\Delta)$ corresponds to a measurement where we take a particle and measure it by asking if it would transist to any of the states $T^\eta(p, q)$ for $(p, q) \in \Delta$. Similarly for $A^\eta(F)$. For example, we next take $B = f(P) = A^\eta(\mathcal{F})$ and $C = j(Q) = A^\eta(\mathcal{J})$ for some $\mathcal{F}$ and $\mathcal{J}$ between 0 and 1. (Note that $\mathcal{F}(p, q) = \mathcal{F}(p, 0)$ and $\mathcal{J}(p, q) = \mathcal{J}(0, q)$.) Thus

$$B \oplus C = A^\eta(\mathcal{F} + \mathcal{J})$$

as long as $(\mathcal{F} + \mathcal{J})(x) \leq 1$. It corresponds to the experiment in which you will describe the particle transisting to the state $T^\eta(p, q)$ located in the fuzzy set $\mathcal{F} + \mathcal{J}$.

## 4. THE SET OF EFFECTS FOR THE FORMALISM OF QUANTUM MECHANICS ON PHASE SPACE AS AN M. V. EFFECT ALGEBRA

While we are considering $\mathcal{E}$ equal to the set of effects for the formalism of quantum mechanics on phase space with respect to $\eta$, we may consider exactly what structure $\mathcal{E}$ has within the chain: effect algebra $\supset$ interpolation algebra $\supset$ Riesz decomposition algebra $\supset$ lattice ordered effect algebra $\supset$ distributive algebra $\supset$ M. V. effect algebra $\supset$ Heyting effect algebra $\supset$ Boolean algebra. [See Foulis (2000) for the definitions.] We have previously shown that $\mathcal{E}$ is an effect algebra (Schroeck, 2005). We will show now that $\mathcal{E}$ satisfies the axioms of an M. V. effect algebra, and in fact a Heyting effect algebra, so that it also satisfies the axioms of all the intermediate algebras as well. It is not a Boolean algebra, as we will show.

*Definition 2.* We, along with C. C. Chang, D. Mundici and D. Foulis, define an M. V. algebra as a set $(M, 0, I, ', \boxplus)$ where $M$ is a set, 0 and $I$ are distinguished elements of $M$, $'$ is a unary operation on $M$, $\boxplus$ is a binary operation on $M$ that is associative and commutative, and follows the axioms

$$a \boxplus 0 = a, a \boxplus I = I, a'' = a, 0' = I, a \boxplus a' = I,$$

and (the axiom of Łukasiewicz)

$$(a \boxplus b')' \boxplus a = (b \boxplus a')' \boxplus b.$$

Then we define $a \leq b \Leftrightarrow b = (a \boxplus b')' \boxplus a$ making $(M, \leq)$ a poset. It is a distributive lattice with $a \vee b = (a \boxplus b')' \boxplus a$, $a \wedge b = (a' \vee b')'$.

*Definition 3.* An M. V. effect algebra is a set $(M, 0, I, \oplus)$, $\oplus$ is a partially defined binary operation on $M$, where $(M, 0, I)$ is lattice ordered, and $a \wedge b = 0 \Rightarrow a \leq b'$.

We then have the following theorem:

**Theorem 1.** *(Foulis, 2000) Every M. V. effect algebra* $(M, 0, I, \oplus)$ *is equivalent to an M. V. algebra* $(M, 0, I, ', \boxplus)$.

Outline of proof: $\Leftarrow$: $a \oplus b$ is defined iff $a \leq b'$.

$\Rightarrow$: We have the lattice properties with the distributive property as well as $'$ defined. So define $a \boxplus b = a \oplus (a' \wedge b)$. Then the rest follow, the associative property being tricky.

We must show that $\mathcal{E}$ is an M. V. effect algebra. We have already defined $\oplus$ and 0 and $I$ are the zero and identity operators. $(A^\eta(f))' = I - A^\eta(f) = A^\eta(1 - f)$ where 1 is the constant function $1(x) = 1$.

*Definition 4*. Let $A^\eta(f)$ and $A^\eta(g) \in \mathcal{E}$. Define

$$A^\eta(f) \wedge A^\eta(g) = A^\eta(\min(f, g))$$

where we mean "min" holds at each $x \in \Gamma$ a.e. $\mu$. Then we also have, with a similar meaning of "max,"

$$A^\eta(f) \vee A^\eta(g) = A^\eta(\max(f, g)) = [A^\eta(f)' \wedge A^\eta(g)']'.$$

Now, using the standard connection between $(\mathcal{E}, \oplus, ', 0, I)$ as an M. V. effect algebra and $(\mathcal{E}, \boxplus, ', 0, I)$ as an M. V. algebra, we compute, for $0 \leq f, g \leq 1$,

$$\begin{aligned} A^\eta(f) \boxplus A^\eta(g) &= A^\eta(f) \oplus [A^\eta(f)' \wedge A^\eta(g)] \\ &= A^\eta(f) \oplus [A^\eta(1-f) \wedge A^\eta(g)] \\ &= A^\eta(f) \oplus A^\eta(\min(1-f, g)) \\ &= A^\eta(f + \min(1-f, g)) \\ &= A^\eta((\min(f+g), 1). \end{aligned}$$

With a slight abuse of notation, we will write this as

$$A^\eta(f) \boxplus A^\eta(g) \equiv A^\eta((f+g) \wedge 1).$$

Then $\boxplus$ is defined everywhere as a binary operation on $\mathcal{E}$ which is commutative and associative, $A^\eta(f) \boxplus 0 = A^\eta(f)$, $A^\eta(f) \boxplus 1 = 1$, $A^\eta(f)'' = A^\eta(f)$, $0' = 1$, and $A^\eta(f) \boxplus A^\eta(f)' = 1$. We next show that the axiom of Łukasiewicz holds:

$$\begin{aligned} [A^\eta(f) \boxplus A^\eta(g)']' \boxplus A^\eta(f) &= [A^\eta(f) \boxplus A^\eta(1-g)]' \boxplus A^\eta(f) \\ &= A^\eta(\{1+f-g\} \wedge 1)' \boxplus A^\eta(f) \\ &= A^\eta(1 - \{1+f-g\} \wedge 1) \boxplus A^\eta(f) \\ &= A^\eta([1+f-\{1+f-g\} \wedge 1] \wedge 1), \end{aligned}$$

and

$$[A^\eta(g) \boxplus A^\eta(f)']' \boxplus A^\eta(g) = A^\eta([1+g-\{1+g-f\} \wedge 1] \wedge 1).$$

But these two expressions are equal, as can be checked for the cases $0 \leq g(x) \leq f(x) \leq 1$ and $0 \leq f(x) \leq g(x) \leq 1$ a.e. We obtain $A^\eta(\max(f, g)) = A^\eta(f) \vee A^\eta(g)$ for either result. Moreover,

$$0 \leq f \leq g \leq 1 \text{ iff } A^\eta(f) \leq A^\eta(g) \text{ iff } A^\eta(g) = [A^\eta(f) \boxplus A^\eta(g)']' \boxplus A^\eta(f).$$

Thus, $(\mathcal{E}, \leq)$ is a distributive lattice. In other words, $\mathcal{E}$ is an M. V. effect algebra.

We have an alternate proof: Consider the map

$$f \mapsto A^\eta(f),$$

$$\text{real-valued } \mu - measurable\ \ functions\ \ on\ \Gamma \text{ between 0 and 1}$$
$$\rightarrow \text{ bounded linear operators on } \mathfrak{H}.$$

Then we have an injection and it suffices to prove that the measurable functions on Γ between 0 and 1 form an M. V. effect algebra, which is well known (Foulis, 2000). ∎

## 5. THE SET OF EFFECTS FOR THE FORMALISM OF QUANTUM MECHANICS ON PHASE SPACE AS A HEYTING ALGEBRA

*Definition 5.* If $P$ is a poset with a 0, then $x \mapsto \tilde{x}$ is a pseudo-complementation iff $x \wedge y = 0 \Leftrightarrow y \leq \tilde{x}$ for $x, y \in P$.

*Definition 6.* A Heyting effect algebra is lattice ordered with a center-valued pseudo-complementation.

We have a theorem that makes this a bit easier to handle:

**Theorem 2.** *If $E$ is an M. V. effect algebra, then the center of $E$, $C(E)$, is given by*

$$\begin{aligned} C(E) &= \{c \in E \mid c \wedge c' = 0\} \\ &= \{c \in E \mid \exists d \in E \text{such that } c \wedge d = 0, c \vee d = I\}. \end{aligned}$$

Since $\mathcal{E}$ is an M. V. effect algebra, the center of $\mathcal{E}$, $C(\mathcal{E})$, is

$$C(\mathcal{E}) = \{c \in \mathcal{E} \mid c \wedge c' = 0\}.$$

But, if $c = A^{\eta}(f)$, then

$$c \wedge c' = A^{\eta}(f) \wedge A^{\eta}(1 - f) = A^{\eta}(\min(f, 1 - f)) = 0$$
$$\text{iff } \min(f, 1 - f) = 0 \text{ a.e.}$$

Thus, $f(x)$ is either 0 or 1 a.e. $x$. Therefore,

$$C(\mathcal{E}) = \{A^{\eta}(f) \in \mathcal{E} \mid f(x) \in \{0, 1\} \text{ a.e. } x\}.$$

Next, consider $A^{\eta}(f) \in \mathcal{E}$, $f$ fixed. Consider $A^{\eta}(g) \in \mathcal{E}$ with $A^{\eta}(f) \wedge A^{\eta}(g) = 0$. Thus $\min(f(x), g(x)) = 0$ a.e. $x$. Take $\tilde{f}$ to be the characteristic function of the complement of supp $(f)$. Then $A^{\eta}(f) \wedge A^{\eta}(g) = 0$ for all $g \leq \tilde{f}$ a.e. Furthermore, there is no other $A^{\eta}(g)$ which has the property $A^{\eta}(f) \wedge A^{\eta}(g) = 0$; so, $A^{\eta}(\tilde{f})$ is the pseudocomplement of $A^{\eta}(f)$. But $A^{\eta}(\tilde{f}) \in C(\mathcal{E})$, and we have shown that $\mathcal{E}$ is a Heyting effect algebra. (We also have an alternate proof, using the fact that we have the injection and using (Foulis, 2000).)

As a consequence, $\mathcal{E}$ is an effect algebra, an interpolation algebra, a Riesz decomposition algebra, a lattice ordered effect algebra, a distributive algebra, (and an M. V. effect algebra).

## 6. $\mathcal{E}$ IS NOT A BOOLEAN ALGEBRA

We take the axioms for a Boolean algebra as well-known. Then we note that $\mathcal{E}$ is not a Boolean algebra, as we have $A^\eta(f) \wedge [A^\eta(f)]' = A^\eta(\min(f, 1 - f)) \neq 0$ for a general $f$.

## 7. PROJECTIONS COMING FROM THE EFFECT ALGEBRA FOR THE FORMALISM OF QUANTUM MECHANICS ON PHASE SPACE

We have previously proved that $\mathcal{E}$ does not contain any projections other than the trivial ones (Schroeck, 1996). We wish to see if there is any way to get various nontrivial projections directly from $\mathcal{E}$.

As an example, we return to the case of spin. With the notation previously established, we have $A^\eta(f(x)) = A^\eta(\frac{1}{2}(1 + 3b \cdot x)) = T(b)$. But $f(x) = \frac{1}{2}(1 + 3b \cdot x)$ is not a fuzzy function. In fact $-1 \leq f(x) \leq 2$. There are other forms of $f$ that will also give $T(b)$, but they are worse in that the corresponding $f$ has a larger range. Consequently, $I$ and 0 are the only projections in $\{A^\eta(f) \mid 0 \leq f(x) \leq 1\}$, consistent with what we have proven before. Nonetheless, we have an effect algebra quite different from the algebra of all effects in $L^2(\mathcal{S}^2)$. Therefore, we have a proper subset of the set of all effects, one that contains no noncommuting projections.

Now in general, if we have informational completeness for the $A^\eta$, as we also do for the nonrelativistic spinless quantum mechanics with $\eta$ equal to a Gaussian for example, we have a curious situation: From the informational completeness, we can approximate every (bounded, self-adjoint) operator by an operator of the form $A^\eta(f)$, for some measurable $f$, in a topology that comes from the trace. (See Healy and Schroeck, 1995; Schroeck, 1996, Chap. III.3.D). But, we can never get all the operators directly from the effect algebra in which $0 \leq f \leq 1$.

Notice that in our definition of $\mathcal{E}$, if you have a function $f$ that is zero except on a set of $\mu$-measure zero, then $A^\eta(f) = 0$. We can make a definition that effectively defines $A(f)$ for a function $f$ that is zero except on a set of $\mu$-measure zero to be nonzero only by enlarging our viewpoint. For this, let $\rho$ be a density operator (that is a trace class operator of trace one) on $L^2(\Gamma, d\mu)$, and let us consider $Tr(\rho A^\eta(f))$. With the definition

$$\rho_{\text{class}}(x) = Tr(\rho T^\eta(x))$$

and recalling that $T^\eta(x) = | \, W(\sigma(x))\eta > \, < W(\sigma(x))\eta \, |$, that $W(g)$ is a unitary continuous representation of $\mathcal{G}$ in $L^2(\Gamma, d\mu)$, and that $\sigma(x)$ is a continuous cross section of $\mathcal{G}$, we have that $\rho_{\text{class}}$ is a continuous function. Moreover, $\rho_{\text{class}}(x)$ is nonnegative, lies between 0 and 1, and integrates to unity. Thus, it is a Kolmogorov probability density. Furthermore, we have

$$Tr(\rho A^\eta(f)) = \int_\Gamma \rho_{\text{class}}(x) f(x) d\mu(x).$$

Thus, we may take an approximate delta funtion $\{f_n(x)\} \to \delta(x - x_o)$ with respect to $\mu$, $f_n \in \{\mu-\text{measurable functions}\}$, $f_n \geq 0$, for the $f$ in this expression. Define $N_n = \text{ess sup } f_n$. Then $N_n^{-1} f_n \in \mathcal{E}$ and

$$N_n Tr\big(\rho A^\eta(N_n^{-1} f_n)\big) = Tr(\rho A^\eta(f_n)) = \int_\Gamma \rho_{\text{class}}(x) f_n(x) d\mu(x)$$
$$\to \rho_{\text{class}}(x_o) = Tr(\rho T^\eta(x_o)).$$

In this fashion, we can get the nontrivial projections $\{T^\eta(x_o) \mid x_o \in \Gamma\}$ from the algebra of effects, $\mathcal{E}$. We also can get, in this way, any sums of the projections in $\{T^\eta(x_o) \mid x_o \in \Gamma\}$, as well as the $A^\eta(f)$ for $f$ any measurable set in $\Gamma$ with measure zero. We can even make a new M. V. algebra from these as well as the $A^\eta(f)$'s. This is equivalent to going from the set of equivalence classes of measurable functions on $\Gamma$ to the set of all measurable functions on $\Gamma$. This is a well defined process on any set and yields an M. V. algebra, for which we can then take the operator equivalent. But, this process being a weak process, we cannot get anything like a product involving these projections and $A^\eta(f)$s.

If $\Gamma$ is a compact set (such as for spin), we can do quite a bit better. Then we have that all the bounded operators are in the set $\{A^\eta(f) \mid f \text{ is measurable}\}$, including all the projections. But we still do not have the nontrivial projections in $\mathcal{E}$.

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